(a) $f(x|\theta,\phi) = \frac{1}{\theta\phi}\left(\frac{x}{\theta}\right)^{\frac{1-\phi}{\phi}}$, $0 \le x \le \theta$, $\theta > 0$, $0 < \phi < 1$

(a) Find a two dimensional sufficient statistic for (θ,ϕ)

The ~~Sample~~ Joint pdf of Sample x is given by:

$$f(x|\theta,\phi) = \prod_{i=1}^{n} \frac{1}{\theta\phi}\left(\frac{x}{\theta}\right)^{\frac{1-\phi}{\phi}}, \quad 0 < x_i < \theta \quad 0 < \phi < 1 \quad i = 1, 2 \cdots n.$$

We can use the indicator function. $I_A(x)$ is the indicator function of the set A; that is, it is equal to 1 if $x \in A$ and equal to 0 otherwise. Now $0 < x_i < \theta$. $i = 1, 2, \cdots n$ if and only iff $\min(x_1, x_2 \cdots x_n\} > \theta$ so we have:

$$f(x|\theta,\phi) = \prod_{i=1}^{n}\left\{\frac{1}{\theta\phi}\left(\frac{x}{\theta}\right)^{\frac{1-\phi}{\phi}}\right\} I_{(0,\theta)}(x_i)$$

$$= \frac{1}{(\theta\phi)^n}\left\{\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\frac{1-\phi}{\phi}}\right\} I_{(-\infty, x_1)}(\theta)$$

$$= \frac{1}{(\theta\phi)^n}\left\{\frac{\sum_{i=1}^{n} x_i}{\theta}^{\, n\left(\frac{1-\phi}{\phi}\right)}\right\} I_{(-\infty, x_1)}\theta$$

$$= \frac{1}{(\theta\phi)^n}\left\{\frac{n-n\phi}{\phi}\right\}\left\{\sum_{i=1}^{n}\frac{x_i}{\theta}\right\} I_{(-\infty, x_1)}\theta$$

Since $f(x|\theta,\phi) = h(x)\, g(T(x)|\theta,\phi)$ where $h(x) = 1$ and

$$g\, T(x)|\theta,\phi) = \frac{1}{(\theta\phi)^n}\left(\frac{n-n\phi}{\phi}\right)\left(\sum_{i=1}^{n}\frac{x_i}{\theta}\right) I_{(-\infty, x_1)}\theta,$$

$$T_1(x) = x_{(1)} \quad T_2(x) = \sum_{i=1}^{n} x_i$$

Therefore, by the factorization Theorem, $(x_{(1)}, \sum_{i=1}^{n} x_i)$ is a a two dimensional sufficient statistic for (θ,ϕ)

#

(b) Show that the sufficient statistic in Part (a) is also minimal sufficient or find a minimal sufficient statistic for (θ, ϕ)

Ans A sufficient statistic $T(x)$ is called a minimal sufficient statistic if, for any other sufficient statistic $T'(x)$, $T(x)$ is a function of $T'(x)$

To say that $T(x)$ is a function of $T'(x)$ simply means that if $T'(x) = T'(y)$ then $T(x) = T(y)$

$$f(x \mid \theta, \phi) = \prod_{i=1}^{n} \frac{1}{\theta\phi} \left(\frac{x}{\theta}\right) \frac{1-\phi}{\phi}$$

$$= \left(\frac{1}{\theta\phi}\right)^n \sum_{i=1}^{n} \frac{x_i}{\theta} \frac{(1-\phi)n}{\phi}$$

Now write

$$\sum_{i=1}^{n} \frac{x_i}{\theta} = \sum_{i=1}^{n} \frac{x_i}{\cancel{\theta}} + n\left(\frac{\bar{x}}{\phi}\right)$$

Form the ratio

$$\frac{f(x \mid \theta, \phi)}{f(y \mid \theta\phi)} = \frac{\left(\frac{1}{\theta\phi}\right)^n \sum_{i=1}^{n} \frac{x_i}{\theta} \, n\left(\frac{1-\phi}{\phi}\right)}{\left(\frac{1}{\theta\phi}\right)^n \sum_{i=1}^{n} \frac{y_i}{\theta} \, n\left(\frac{1-\phi}{\phi}\right)}$$

$$= \frac{\left(\frac{1}{\theta\phi}\right)^n \sum_{i=1}^{n} \frac{x_i}{\bar{x}} + n\left(\frac{\bar{x}}{\phi}\right)}{\left(\frac{1}{\theta\phi}\right)^n \sum_{i=1}^{n} \frac{y_i}{\bar{y}} + n\left(\frac{\bar{y}}{\phi}\right)}$$

Clearly, this ratio is free of θ if and only if $\bar{x} = \bar{y}$ we know that $T(x) = \bar{x}$ is a minimal sufficient statistic.

(c) Show that the density $T = X_n$ is given by

$$f X_n = \begin{cases} \frac{n}{\theta\phi}\left(\frac{x}{\theta}\right)^{\frac{n}{\phi}-1} & \text{for } 0 \le x \le \theta \quad \text{~~otherwise~~ zero.} \\ 0 & \text{other wise} \end{cases}$$

By factorization theorem X_n is sufficient for θ

X_n has a pdf

$$f_{X_n}(x) = \left\{ \frac{n}{\theta\phi}\left(\frac{x}{\theta}\right)^{\frac{n}{\phi}-1}\right.$$

For any g,

$$E_\theta\{g(X_n)\} = \frac{n}{\theta\phi}\int_0^\theta g(x)\left(\frac{x}{\theta}\right)^{\frac{n}{\phi}-1} dx = 0 \quad \text{all } \theta > 0$$

implies that

$$0 = \int_0^\theta g(x)\left(\frac{x}{\theta}\right)^{\frac{n}{\phi}-1} dx \text{ and } 0 = \frac{d}{d\theta}\int_0^\theta g(x)\left(\frac{x}{\theta}\right)^{\frac{n}{\phi}-1} dx$$

$$\cancel{= g(\theta)\,\theta^{n-1}} \quad \text{all } \theta > 0$$

$$= g(\theta)\left(\frac{1}{\theta}\right)^{\frac{n}{\phi}-1}$$

Thus $g(x) = 0$ for all $x > 0$ which means $P_\theta(g(X) = 0) = 1$ for all θ. Hence X_n is also complete, Therefore

$T = X_n$ is given by

$$f X_n = \begin{cases} \frac{n}{\theta\phi}\left(\frac{x}{\theta}\right)^{\frac{n}{\phi}-1} \\ 0 & \text{otherwise} \end{cases}$$

#

(d) Show that X_n is sufficient and complete

A statistic $T(X)$ is sufficient for θ iff the conditional distribution of X given $T=t$ does not depend on θ.

A statistic $T(X)$ is complete iff for any g not depending on θ $E_\theta[g(X)]=0$ for all $\theta \in \Theta$ implies $P_\theta(g(X)=0) = 1 \; \forall \theta \in \Theta$

$$0 = E_\theta[g(x)] = \sum_{x=0}^{n} g(x) \frac{1}{\theta\phi} \left(\frac{x}{\theta}\right)^{\frac{1-\phi}{\phi}}$$

$$= \cancel{\left(\frac{1-\phi}{\phi}\right)^{n}}$$

$$= \left(\frac{1}{\theta\phi}\right)^n \sum_{x=0}^{n} \left(\frac{x}{\theta}\right)^{\frac{1-\phi}{\phi}} \quad \text{all } \theta \in (0,\theta)$$

Since the factor $\left(\frac{1}{\theta\phi}\right)^n \neq 0$ we must have

$$0 = \sum_{x=0}^{n} g(x) \left(\frac{x}{\theta}\right) \varphi^{x} \quad \text{all } \varphi > 0 = \frac{1}{\theta\phi}$$

- The last expression is a polynomial φ of ~~degree~~ degree n.
- For this polynomial to be 0 for all $\varphi > 0$ it must be true that the coefficient of φ^x, which is $g(x)\frac{x}{\theta}$ is 0 for every x.
- This show that $g(x)=0$ for $x=0$ for $x=0,\dots,n$ and hence $P_\theta(g(X)=0)=1$ for all θ. Therefore X is sufficient and complete

#/

(e) Otherwise Compute the UMVUE of ϕ

$$f(x;\theta,\phi) = \frac{1}{\theta\phi}\left(\frac{x}{\theta}\right)^{\frac{1-\phi}{\phi}}$$

Suppose $V = \theta$

Since the sufficient and complete statistic X_n has the Lebesgue p.d.f

$$\left(\frac{1}{\theta\phi}\right)^n \left\{\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\frac{1-\phi}{\phi}}\right\} I_{(0,\theta)}\, x_i$$

$$E X_{(n)} = \left(\frac{1}{\theta\phi}\right)^n \int_0^\theta \frac{x_i}{\theta}\, dx = \frac{n}{n+1}\theta$$

Hence the unbiased estimator of θ is $(n+1)X_n/n$, which is the UMVUE.

Suppose that $V = g(\theta)$, where g is a differentiable function on $(0,\theta)$

An unbiased estimator $h(X_{(n)})$ of V must satisfy

$$\theta^n g(\theta) = n\int_0^\theta h(x)\, x^{\frac{1-\phi}{\phi}}\, dx \quad \text{for all } \theta > 0$$

Differentiating both sizes of the previous equation and applying the result of differentiation of an integral lead to

$$n\theta^{n-1} g(\theta) + \theta^n g'(\theta) = n\, h(\theta)\, \theta^{n-1}$$

Hence the UMVUE of V is $h(X_{(n)}) = g(X_{(n)}) + n^{-1} X_{(n)} g'(X_{(n)})$

In particular, if $V = \theta$, then UMVUE of θ is $(1+n^{-1})X_n$.